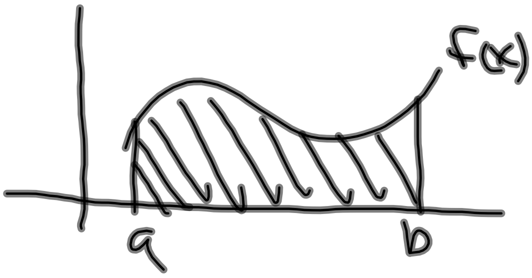
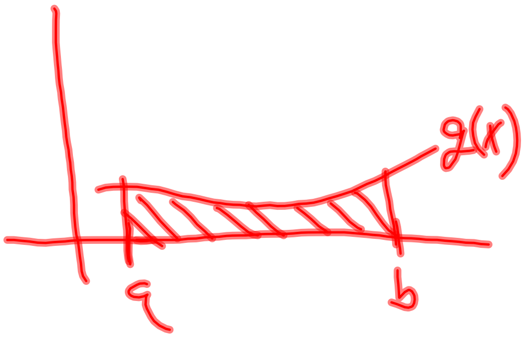
Disks, Washers, Shells, Slices

this is not a building materials course, these are *applications of the definite integral.*

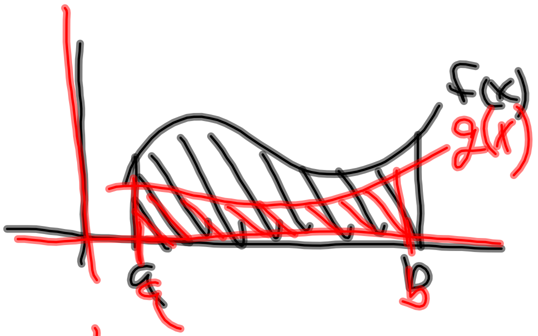
Area between two curves



$$= \int_a^b f(x)\,dx$$



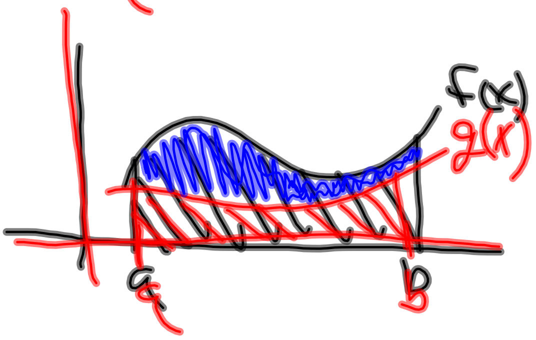
$$= \int_a^b g(x)\,dx$$



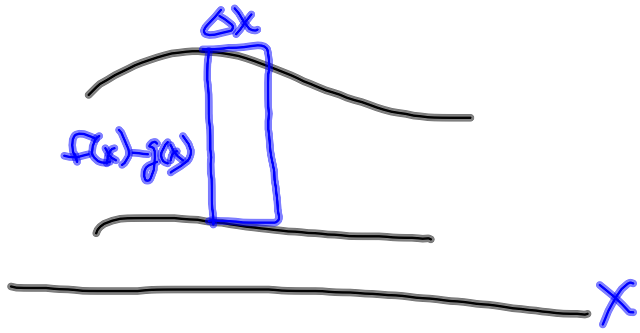
$A = A - A$ right?

$A = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$

$= \int_a^b (f(x) - g(x))\,dx$



$g(x) \leq f(x)$ $[a,b]$

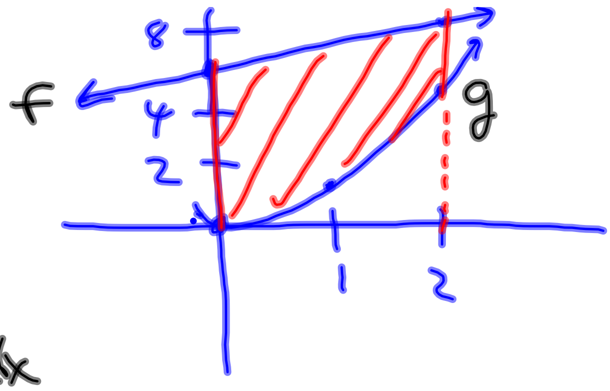


Δx
f(x)-g(x)
x

$y = x + 6$

$y = x^2$

$[0, 2]$

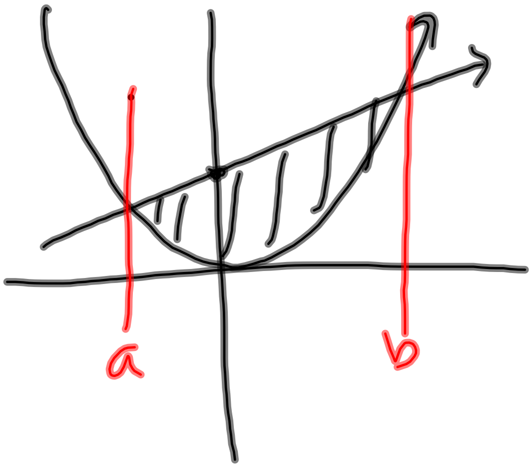


$$A = \int_0^2 \left[(x+6) - x^2\right] dx$$

$$A = \int_0^2 (-x^2 + x + 6)\, dx = -\frac{x^3}{3} + \frac{x^2}{2} + 6x \Big|_0^2 = -\frac{8}{3} + 2 + 12 - 0$$

$$= \frac{34}{3}$$

$y = x^2$
$y = x + 6$ } area between



$x^2 = x + 6$

$x^2 - x - 6 = 0$

$(x+2)(x-3) = 0$

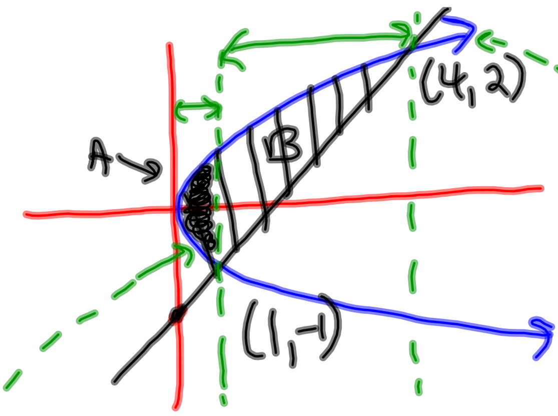
$x = -2 \quad x = 3$

$a \quad b$

$$\int_{-2}^{3} (x + 6 - x^2)\, dx$$

$$\boxed{\begin{aligned} x &= y^2 \\ x &= y+2 \end{aligned}}$$ area between

$y = x - 2$



$y^2 = y + 2$

$y^2 - y - 2 = 0$

$(y-2)(y+1) = 0$

$y = 2, -1$

$x = 4$

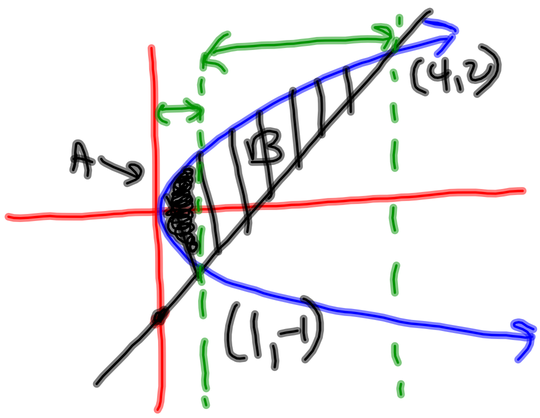
$x = 1$

notice $x = y^2$

$\Rightarrow y = \pm\sqrt{x}$ we care about $y = \sqrt{x}$

$y = -\sqrt{x}$

$$A = \int_0^1 \left(\sqrt{x} - (-\sqrt{x})\right) dx = 2\int_0^1 \sqrt{x}\, dx = 2 \left. \frac{x^{3/2}}{\frac{3}{2}} \right|_0^1 = \frac{4}{3}(1) = \frac{4}{3}$$



$A = \frac{4}{3}$

$B = \int_1^4 \left[\sqrt{x} - (x-2)\right] dx$

$= \int_1^4 (-x + \sqrt{x} + 2)\, dx$

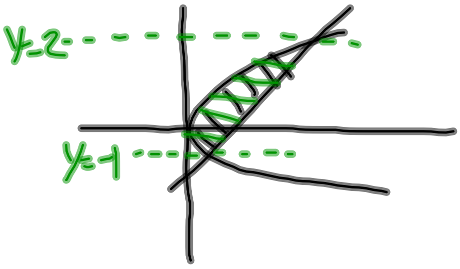
$= -\frac{x^2}{2} + \frac{x^{3/2}}{\frac{3}{2}} + 2x \Big|_1^4$

$= -8 + \frac{16}{3} + 8 - \left(-\frac{1}{2} + \frac{2}{3} + 2\right) = \frac{19}{6}$

$\text{Area} = \frac{4}{3} + \frac{19}{6} = \frac{27}{6} = \frac{9}{2}$

$$\boxed{\begin{array}{l} x = y^2 \\ x = y + 2 \end{array}}$$ area between

$y = x - 2$



$$A = \int_{-1}^{2} \left((y+2) - y^2 \right) dy$$

$$= \int_{-1}^{2} \left(-y^2 + y + 2 \right) dy = -\frac{y^3}{3} + \frac{y^2}{2} + 2y \Big|_{-1}^{2}$$

$$= -\frac{8}{3} + 2 + 4 - \left(\frac{1}{3} + \frac{1}{2} - 2 \right) = \frac{10}{3} - -\frac{7}{6}$$

$$= \frac{27}{6} = \frac{9}{2}$$

First, take a look at the animation that you are sooooo lucky to have! When I was a kid :) we had to just "imagine" this!

http://curvebank.calstatela.edu/volrev/volrev.htm

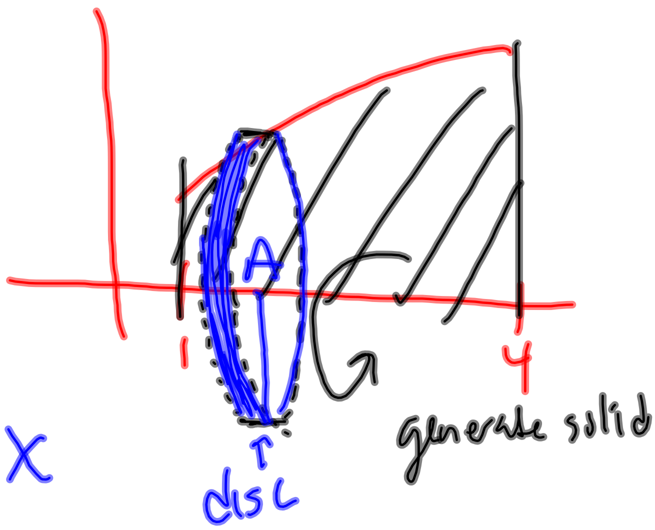
(after much discussion)

$$A = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k^*)\Delta x = \int_a^b f(x)\,dx$$

$$V = \lim_{n\to\infty} \sum_{k=1}^{n} A(x_k^*)\Delta x = \int_a^b A(x)\,dx$$

figure out a formula for the area of the surface of a slice, and multiply it by the thickness of the slice to get the voulme. Then add them all up (integerate)

$y=\sqrt{x}$ [1,4]



area of disc

πr^2

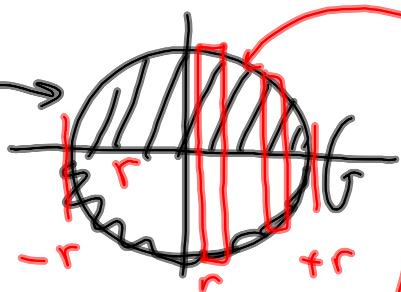
$\pi (f(x))^2 = \pi(\sqrt{x})^2 = \pi x$

thickness of disc = Δx

$$V = \int_1^4 (\pi x)\,dx = \pi \frac{x^2}{2}\Big|_1^4 = \pi\left(\frac{16}{2}-\frac{1}{2}\right) = \frac{15\pi}{2} \doteq 23.6$$

$$x^2+y^2=r^2$$

$$y=\sqrt{r^2-x^2}$$



this r changes

sphere

$$\int_{-r}^{+r}\pi r^2\,dx = 2\int_0^r \pi r^2\,dx = 2\pi\int_0^r \left(\sqrt{r^2-x^2}\right)^2 dx$$

$$=2\pi\int_0^r (r^2-x^2)\,dx = 2\pi\left(r^2x-\frac{x^3}{3}\right)\Big|_0^r = 2\pi\left(r^3-\frac{r^3}{3}-0\right)$$

$$=2\pi\left(\frac{2}{3}r^3\right)=\frac{4}{3}\pi r^3$$

visual for "washer" method:

http://mathdemos.org/mathdemos/washermethod/gallery/gallery.html

$y = \frac{1}{2} + x^2$

$y = x$

$A = \pi r_o^2 - \pi r_i^2$

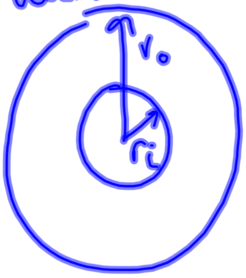
$= \pi(r_o^2 - r_i^2)$

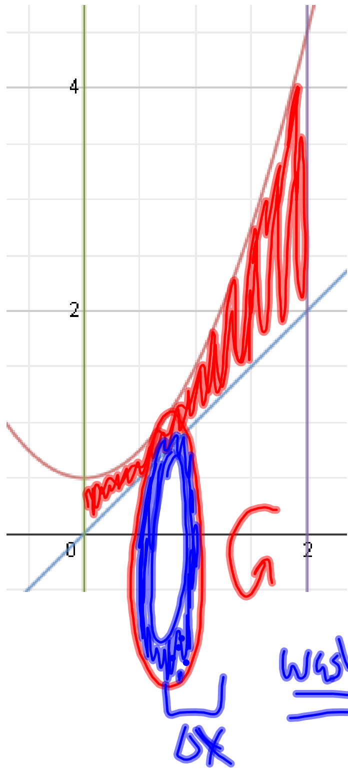
$= \pi\left[\left(\frac{1}{2} + x^2\right)^2 - x^2\right]$

$= \pi\left(\frac{1}{4} + \cancel{x^2} + x^4 - \cancel{x^2}\right)$

$A = \pi\left(\frac{1}{4} + x^4\right)$

washer





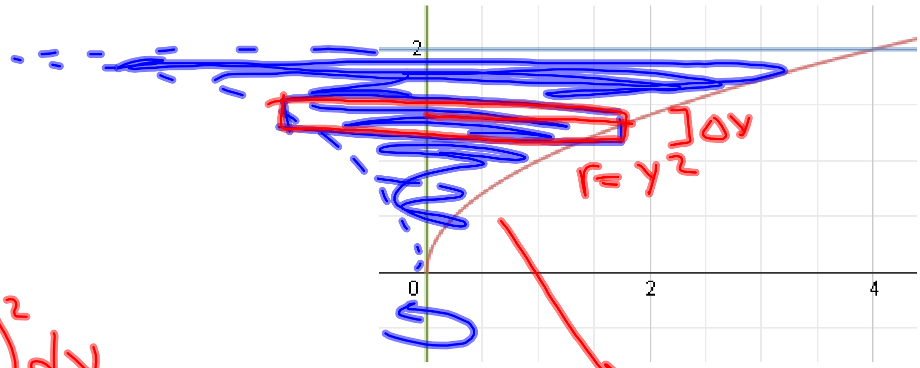
washer

$$A = \pi\left(\frac{1}{4} + x^4\right)$$

$$V = \pi\int_0^2 \left(\frac{1}{4} + x^4\right)dx$$

$$V = \pi\left(\frac{1}{4}x + \frac{x^5}{5}\right)_0^2 = \pi\left(\frac{1}{2} + \frac{32}{5} - 0\right) = \frac{69\pi}{10}$$

$y = \sqrt{x}$
$y = 2$
$x = 0$



$$V = \int_0^2 \pi (y^2)^2\,dy$$

$$= \pi \int_0^2 y^4\,dy = \pi \left.\frac{y^5}{5}\right|_0^2 = \frac{32\pi}{5}$$

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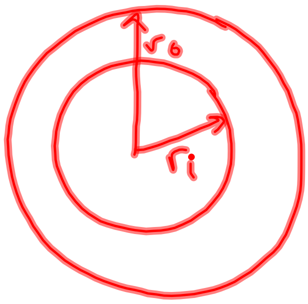
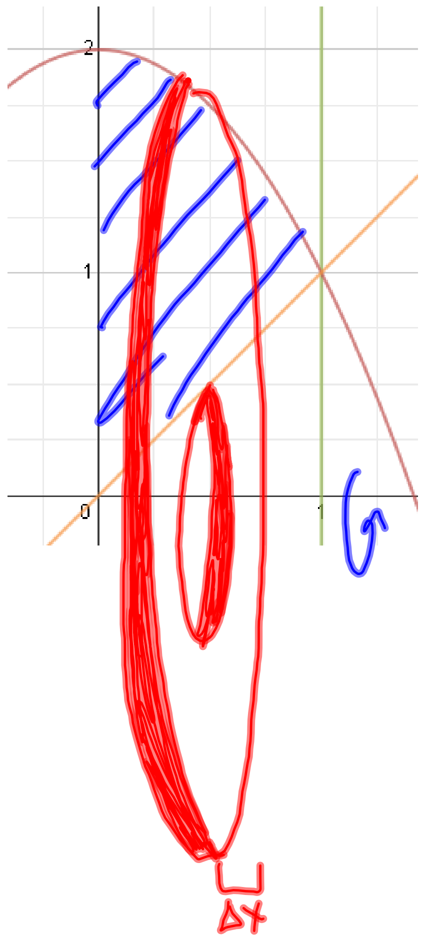
$y = x$

$y = 2 - x^2$



$A = \pi r_o^2 - \pi r_i^2$



$$V = \pi \int_0^1 \left[(2-x^2)^2 - (x^2)\right] dx$$

$$= \pi \int_0^1 (4 - 4x^2 + x^4 - x^2)\, dx$$

$$= \pi \int_0^1 (4 - 5x^2 + x^4)\, dx$$

$$= \pi \int_0^1 \left(4 - 5x^2 + x^4\right) dx$$

$$\pi\left(4x - \frac{5}{3}x^3 + \frac{x^5}{5}\right)\Bigg|_0^1 = \pi\left(4 - \frac{5}{3} + \frac{1}{5}\right) = \frac{38\pi}{15}$$

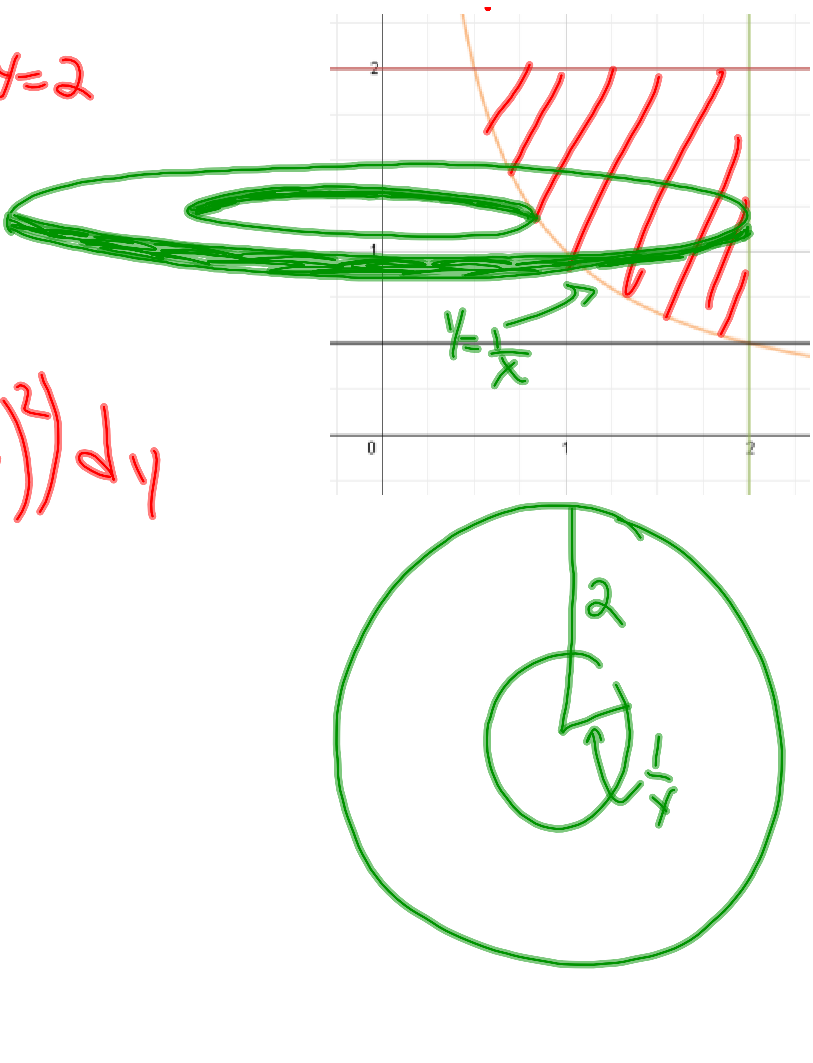
$y = \frac{1}{x}$ $\quad y = \frac{1}{2} \rightarrow y = 2$

$x = \frac{1}{y}$

radius $= \frac{1}{y}$

$$V = \int_{\frac{1}{2}}^{2} \left(\pi (2)^2 - \pi \left(\frac{1}{y}\right)^2 \right) dy$$

$$V = \pi \int_{\frac{1}{2}}^{2} \left(4 - \frac{1}{y^2} \right) dy$$



$$V = \pi \int_{\frac{1}{2}}^{2} \left(4 - \frac{1}{y^2}\right) dy = \pi \left(4y - \frac{y^{-1}}{-1}\right)\Big|_{\frac{1}{2}}^{2} = \pi \left(4y + \frac{1}{y}\right)\Big|_{\frac{1}{2}}^{2}$$

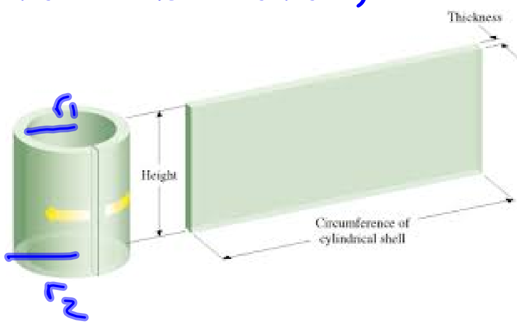
$$= \pi\left[\left(8 + \frac{1}{2}\right) - \left(2 + 2\right)\right] = \pi\left(\frac{17}{2} - 4\right) = \frac{9\pi}{2}$$

Cylindrical Shells

visual: http://www.youtube.com/watch?v=DaUYqq2uUxE

different/better visuals:

http://mathdemos.org/mathdemos/shellmethod/gallery/gallery.html volume of a "shell" is the area (outside circle - inside circle) of the cross section, times the height.



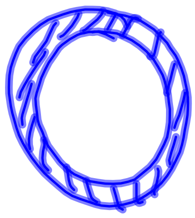
$$A_{\text{"top"}} = \pi r_2^2 - \pi r_1^2$$

$$V = (\pi r_2^2 - \pi r_1^2)h$$

$$= \pi h(r_2^2 - r_1^2)$$

$$= 2\pi h(r_2 - r_1)\frac{(r_2 + r_1)}{2}$$

$$= 2\pi h(\text{thickness})(\text{avg radius})$$



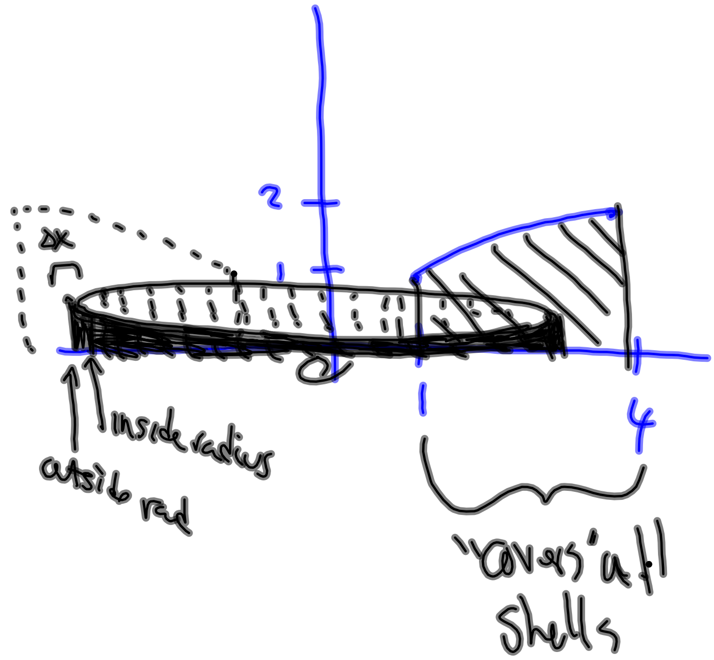
$$V_{shell} = 2\pi h(\text{thickness})(\text{avg radius})$$

$$= 2\pi f(x)(\Delta x)(x_k^*)$$

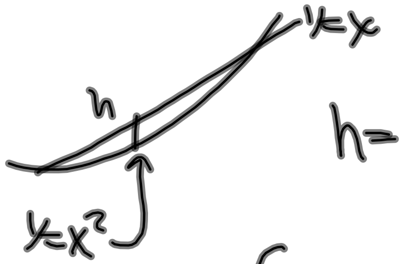
$$\lim_{\Delta x_{max} \to 0} \sum_{k=1}^{n} 2\pi x_k^* f(x) \Delta x = \int_a^b 2\pi x f(x)\, dx = \text{Volume of Solid}$$

$$2\pi \int_a^b x f(x)\, dx$$

ex 1



<u>ex2</u> — height of shell is not just f(x)



$$h = x - x^2$$

$$2\pi \int x(x-x^2)\,dx$$

x "f(x)"

look at the solid generated and convince yourself that you can use washers and integrate with respect to y. You can even actually do it if you are bored.

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$$\sqrt{4-x^2} = x$$
$$4-x^2 = x^2$$
$$4 = 2x^2$$
$$x = \sqrt{2}$$

$$h = \sqrt{4-x^2} - x$$

$$V = \int_0^{\sqrt{2}} 2\pi x\left(\sqrt{4-x^2} - x\right)dx$$

$$V = 2\pi\left(-\frac{1}{2}\int_0^{\sqrt{2}} \sqrt{4-x^2}(-2x\,dx) - \int_0^{\sqrt{2}} x^2\,dx\right)$$

$$= 2\pi\left(-\frac{1}{2}\frac{(4-x^2)^{3/2}}{3/2} - \frac{x^3}{3}\right)\Bigg|_0^{\sqrt{2}}$$

$$= 2\pi\left(-\frac{1}{3}(4-x^2)^{3/2} - \frac{x^3}{3}\right)\Bigg|_0^{\sqrt{2}} = 2\pi\left[\left(-\frac{1}{3}\sqrt{8} - \frac{2\sqrt{2}}{3}\right) - \left(-\frac{8}{3}\right)\right]$$

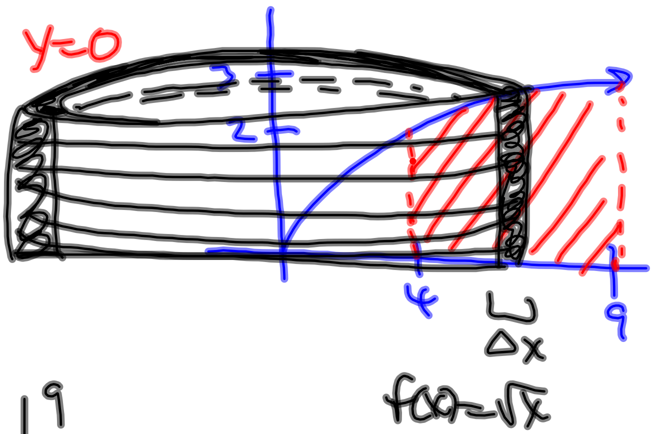
$$2\pi\left[\left(-\frac{1}{3}\sqrt{8}-\frac{2\sqrt{2}}{3}\right)-\left(-\frac{8}{3}\right)\right]$$

$$2\pi\left(-\frac{2\sqrt{2}}{3}-\frac{2\sqrt{2}}{3}+\frac{8}{3}\right)$$

$$2\pi\left(\frac{8-4\sqrt{2}}{3}\right)$$

$$8\pi\left(\frac{2-\sqrt{2}}{3}\right)$$

#6 $y=\sqrt{x}$ $x=4$ $x=9$ $y=0$



$$2\pi\int_4^9 x\sqrt{x}\,dx$$

$$=2\pi\int_4^9 x^{3/2}\,dx = 2\pi\,\frac{2}{5}x^{5/2}\Big|_4^9$$

$$\frac{4\pi}{5}\left(3^5-2^5\right) = \frac{4\pi}{5}(243-32) = \frac{844\pi}{5}$$

P. 467 #1,7,13

P. 473 #7,11,15,19,27*

P. 479 #3,7,15